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A Study of Some Interpolatory Processes Based on the Roots of Legendre Polynomials

J. PRASAD

Department of Mathematics, California State University, Los Angeles, California 90032

AND

A. K. VARMA

Department of Mathematics, University of Florida, Gainesville, Florida 32611

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1. BEHAVIOR OF LAGRANGE AND HERMITE INTERPOLATION ON THE ROOTS OF LEGENDRE POLYNOMIALS

It is well known that the Lagrange interpolation procedure cannot be uniformly convergent for all continuous functions no matter what matrix of nodes of interpolation is chosen. However, L. Fejér [6] proved that for certain special matrices, the Hermite–Fejér interpolation parabolas $H_n(f)$ of any continuous function f on $[-1, 1]$ converge uniformly to f on $[-1, 1]$; e.g., the matrix T, the nth row of which consists of the n roots of $T_n(x)$ (the Tchebycheff polynomial of degree n) displays this property. Fejér also proved that $H_n(f)$ based on the roots of the Legendre polynomials converges uniformly to f in each closed subinterval of $(-1, 1)$. Furthermore, for the endpoints ± 1 he showed that

$$\lim_{n\to\infty} H_n[f, \pm 1] = \tfrac{1}{2}\int_{-1}^{1} f(x)\,dx.$$

For further details in this direction we refer to the interesting work of Szabadös [11].

2. SOME INTERPOLATORY PROCESSES

Let us denote by

$$-1 = x_n < x_{n-1} < \cdots < x_2 < x_1 = 1 \tag{2.1}$$

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the n distinct zeros of $(1-x^2)P_{n-2}(x)$, where $P_n(x)$ is the Legendre polynomial of degree n with the normalization

$$P_n(1)=1. \tag{2.2}$$

We set

$$l_k(x)=\frac{P_{n-2}(x)}{(x-x_k)P'_{n-2}(x_k)}, \qquad k=2,3,\ldots,n-1, \tag{2.3}$$

$$h_1(x)=\frac{1+x}{2}P^2_{n-2}(x), \qquad h_n(x)=\frac{1-x}{2}P^2_{n-2}(x), \tag{2.4}$$

and

$$h_k(x)=\frac{1-x^2}{1-x_k^2}l_k^2(x), \qquad \sigma_k(x)=(x-x_k)h_k(x), \qquad 2\leqslant k\leqslant n-1. \tag{2.5}$$

Let f be a continuous function on $[-1,1]$. We consider the following interpolation processes based on the roots (2.1):

$$A_n[f,x]=\sum_{k=1}^{n} f(x_k)h_k(x) \tag{2.6}$$

and

$$B_n[f,x]=\sum_{k=1}^{n} f(x_k)h_k(x)+\sum_{k=2}^{n-1}\mu'_n(x_k)\sigma_k(x), \tag{2.7}$$

where $\mu_n(x)$ is an algebraic polynomial of degree $\leqslant n$ satisfying

$$|f(x)-\mu_n(x)|\leqslant c_0\omega_2(f,\sqrt{1-x^2}/n). \tag{2.8}$$

$\omega_2(f,\delta)$ is the modulus of smoothness of order 2 of f. Inequanlity (2.8) is an important result due to DeVore [3].

The polynomials $A_n[f]$ were first constructed by Egervary and Turán [4] as the solution of the problem of most economial process.

The polynomials $B_n[f]$ were initiated by Fejér [7] and Szász [12]. It is easy to see that

$$\begin{aligned} A_n[f,x_i]&=f(x_i), \qquad i=1,2,\ldots,n,\\ A'_n[f,x_i]&=0, \qquad i=2,3,\ldots,n-1, \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} B_n[f,x_i]&=f(x_i), \qquad i=1,2,\ldots,n,\\ B'_n[f,x_i]&=\mu'_n(x_i), \qquad i=2,3,\ldots,n-1. \end{aligned} \tag{2.10}$$

3. MAIN TESULT

Concerning $A_n[f]$ and $B_n[f]$ we shall prove the pointwise estimates in the form of the following theorem.

THEOREM 3.1. *Let $f \in C[-1, 1]$ then for $-1 \leqslant x \leqslant 1$ we have*

$$|A_n[f, x] - f(x)| \leqslant c_1 n^{-1} \sum_{i=1}^{n} \omega(f, \sqrt{1-x^2}/i) \tag{3.1}$$

and

$$|B_n[f, x] - f(x)| \leqslant c_2 \omega_2(f, \sqrt{1-x^2}/n), \tag{3.2}$$

where c_1 and c_2 are positive constants independent of f, n and x.

Inequality (3.1) is analogous to the results of Bojanic [2] and Vertesi [14]. Inequality (3.2) is analogous to a recent theorem of DeVore [3]. We note that $B_n[f, x]$ is also interpolatory.

4. PRELIMINARIES

We need some known facts about Legendre polynomials. From [4] we have

$$\sum_{k=2}^{n-1} h_k(x) \equiv 1 - P_{n-2}^2(x) \leqslant 1. \tag{4.1}$$

According to Bernstein [1],

$$(1-x^2)^{1/4} |P_{n-2}(x)| \leqslant \sqrt{2/\pi(n-2)}, \qquad n \geqslant 4. \tag{4.2}$$

From a theorem of Erdös [5] it follows that there exists a $c_3 > 0$ (independent of n and x) such that for $-1 \leqslant x \leqslant 1$,

$$|l_k(x)| \leqslant c_3, \qquad k = 2, 3,..., n-1. \tag{4.3}$$

Recalling the definition of $x_k = \cos \theta_k$ we obtain

$$1 - x_k^2 > (k - \tfrac{3}{2})^2 n^{-2}, \quad k = 2, 3,..., [(n-2)/2], \tag{4.4}$$

and

$$|P'_{n-2}(x_k)| > c_4 (k - \tfrac{3}{2})^{-3/2} n^2, \qquad k = 2, 3,..., [(n-2)/2]. \tag{4.5}$$

We note that a similar estimate holds for $k = [(n-2)/2] + 1, \ldots, n-1$. On combining (4.4) and (4.5) it follows that

$$(1-x_k^2)^{3/4}\,|P'_{n-2}(x_k)| \geqslant c_5 n^{1/2}, \qquad k = 2, 3, \ldots, n-1. \tag{4.6}$$

From (4.2) and (4.7) it follows that

$$\frac{(1-x^2)^{1/4}\,|P_{n-2}(x)|}{(1-x_k^2)^{3/4}\,|P'_{n-2}(x_k)|} \leqslant \frac{c_6}{n}, \qquad -1 \leqslant x \leqslant 1, \quad n \geqslant 4. \tag{4.7}$$

We also see that

$$\sin\theta_k \leqslant \sin\theta + \sin\theta_k \leqslant 2\sin(\theta+\theta_k)/2. \tag{4.9}$$

5. Some Lemmas

Throughout this paper we assume x_j to be that zero of $P_{n-2}(x)$ which is nearest to x. Using the definition of x_j and (4.4) it follows that for some $c_7 > 0$ independent of n and x,

$$\frac{1}{|\sin(\theta-\theta_k)/2|} \leqslant c_7 n (r - \tfrac{1}{2})^{-1},$$
$$k = j \pm r, \qquad r = 1, 2, \ldots, n-3. \tag{5.1}$$

We now prove the following lemmas.

Lemma 5.1. *For* $-1 \leqslant x \leqslant 1$ *we have*

$$|f(1) - f(x)|\, h_1(x) \leqslant c_8\,\omega(\sqrt{1-x^2}/n), \tag{5.2}$$
$$|f(-1) - f(x)|\, h_n(x) \leqslant c_9\,\omega(\sqrt{1-x^2}/n) \tag{5.3}$$

and

$$I_1 \equiv \sum_{k=2}^{n-1} |f(x_k) - f(x)|\, h_k(x) \leqslant c_{10} \sum_{r=1}^{n} \frac{1}{r^2}\,\omega\left(\frac{r\sin\theta}{n}\right), \tag{5.4}$$

where $\omega(\delta) \equiv \omega(f, \delta)$.

Proof. For $x = \pm 1$ (5.2) holds trivially. On using the properties of modulus of continuity of f we have for $-1 < x < 1$,

$$\begin{aligned}|f(1)-f(x)|\, h_1(x) &\leqslant h_1(x)\,\omega(1-x)\\ &\leqslant \left(1+\frac{n\sqrt{1-x}}{\sqrt{1+x}}\right) h_1(x)\,\omega\left(\frac{\sqrt{1-x^2}}{n}\right)\\ &= \left(1+\frac{n\sqrt{1-x}}{\sqrt{1+x}}\right)\frac{(1+x)}{2} P_{n-2}^2(x)\,\omega\left(\frac{\sqrt{1-x^2}}{n}\right)\\ &\leqslant (1+n\sqrt{1-x^2})\, P_{n-2}^2(x)\,\omega\left(\frac{\sqrt{1-x^2}}{n}\right)\\ &\leqslant (1+n\sqrt{1-x^2}\, P_{n-2}^2(x))\,\omega\left(\frac{\sqrt{1-x^2}}{n}\right).\end{aligned}$$

On using (4.2) we obtain (5.2). Proof of (5.3) can be obtained along the same lines. Now we again note that for $x = \pm 1$ (5.4) follows obviously. For $-1 < x < 1$, we divide the sum I according to the definition of x_j as given above. We write

$$I_1 = \sum_{k \neq j} |f(x_k) - f(x)|\, h_k(x) + |f(x_j) - f(x)|\, h_j(x). \tag{5.5}$$

Again, making use of the properties of modulus of continuity of f we obtain

$$\begin{aligned} I_1 &\leqslant \sum_{\substack{k \neq j\\ 2 \leqslant k = j \pm r \leqslant n-1}} \omega(|x - x_k|)\, h_k(x) + h_j \omega(|x - x_j|)\\ &\leqslant \sum_{\substack{k \neq j\\ 2 \leqslant k = j \pm r \leqslant n-1}} \left(1 + \frac{n\,|x - x_k|}{r \sin\theta}\right) \omega\left(\frac{r \sin\theta}{n}\right) h_k(x) \\ &\quad + \left(1 + \frac{n\,|x - x_j|}{\sin\theta}\right) h_j(x)\, \omega\left(\frac{\sin\theta}{n}\right). \end{aligned} \tag{5.6}$$

Further we note that

$$n\,|x - x_k|\, h_k(x) \leqslant c_{11} \sqrt{1 - x^2} \tag{5.7}$$

and

$$n\,|x - x_k|\, h_k(x) \leqslant c_{12} \sqrt{1 - x^2}/r, \qquad k = j \pm r, \quad k \neq j. \tag{5.8}$$

First we shall prove (5.7). From (4.3) and (4.1) it follows that

$$\sum_{k=2}^{n-1} \frac{1 - x^2}{1 - x_k^2}\, l_k^4(x) \leqslant c_3^2 \sum_{k=2}^{n-1} \frac{1 - x^2}{1 - x_k^2}\, l_k^2(x) \leqslant c_3^2. \tag{5.9}$$

Hence for $-1 \leqslant x \leqslant 1$,

$$[(1-x^2)^{1/4}/(1-x_k^2)^{1/4}]\,|l_k(x)| \leqslant c_3^{1/2}, \qquad k=2,3,\ldots,n-1. \tag{5.10}$$

Thus on using (4.7) and (5.10) we obtain

$$\begin{aligned} n\,|x-x_k|\,h_k(x) &= n(1-x^2)^{1/2}\left[\frac{(1-x^2)^{1/4}\,|l_k(x)|}{(1-x_k^2)^{1/4}}\right]\left[\frac{(1-x^2)^{1/4}\,|P_{n-2}(x)|}{(1-x_k^2)^{3/4}\,|P'_{n-2}(x_k)|}\right] \\ &\leqslant n(1-x^2)^{1/2}\,c_3^{1/2}c_6 n^{-1} \\ &\leqslant c_{11}(1-x^2)^{1/2}. \end{aligned}$$

This completes the proof of (5.7) for $-1 \leqslant x \leqslant 1$ and $k=2,3,\ldots,n-1$. In order to prove (5.8) we use (5.1) and (4.8) and observe that for $k \neq j$,

$$\begin{aligned} &\frac{(1-x^2)^{1/4}\,|l_k(x)|}{(1-x_k^2)^{1/4}} \\ &\quad = \frac{(1-x^2)^{1/4}\,|P_{n-2}(x)|}{(1-x_k^2)^{3/4}\,|P'_{n-2}(x_k)|}\left[\frac{\sin\theta_k}{2\left|\sin\dfrac{\theta+\theta_k}{2}\right|\left|\sin\dfrac{\theta-\theta_k}{2}\right|}\right] \\ &\quad \leqslant c_6 n^{-1} c_7 n (r-\tfrac{1}{2})^{-1} \\ &\quad \leqslant c_{13}/r, \qquad k=j\pm r, \quad k\neq j. \end{aligned} \tag{5.11}$$

Now we can see that (5.8) follows from (5.11) and (4.7) as follows:

$$\begin{aligned} n\,|x-x_k|\,h_k(x) &= (1-x^2)^{1/2}\left[\frac{(1-x^2)^{1/4}\,|l_k(x)|}{(1-x_k^2)^{1/4}}\right]\left[\frac{n(1-x^2)^{1/4}\,|P_{n-2}(x)|}{(1-x_k^2)^{3/4}\,|P'_{n-2}(x_k)|}\right] \\ &\leqslant c_{12}\frac{\sqrt{1-x^2}}{r}, \qquad k=j\pm r, \quad k\neq j. \end{aligned}$$

From an earlier result of [8, Lemma 2, p. 277] (also [9, p. 128]),

$$h_k(x) \leqslant c_{14}/r^2, \qquad k=j\pm r, \quad -1 \leqslant x \leqslant 1. \tag{5.12}$$

Thus from (5.6), (5.7), (5.8) and (5.12) we immediately obtain

$$I_1 \leqslant c_{10}\sum_{r=1}^{n}\frac{1}{r^2}\,\omega\left(\frac{r\sin\theta}{n}\right).$$

This completes the proof of Lemma 5.1.

For the proof of the inequality (3.2) we need the following lemma.

LEMMA 5.2. *For* $-1 \leqslant x \leqslant 1$ *we have*

$$I_2 = \sum_{k=2}^{n-1} h_k(x)\, \omega_2\left(\frac{\sqrt{1-x_k^2}}{n}\right) \leqslant c_{15}\, \omega_2\left(\frac{\sqrt{1-x^2}}{n}\right). \tag{5.13}$$

Proof. Due to the properties of modulus of continuity of order 2 of $f(x)$ and (4.1) it follows that for $-1 < x < 1$,

$$\begin{aligned} I_2 &\leqslant \sum_{k=2}^{n-1} h_k(x) \left(1 + \frac{\sqrt{1-x_k^2}}{\sqrt{1-x^2}}\right) \omega_2\left(\frac{\sqrt{1-x^2}}{n}\right) \\ &\leqslant \omega_2\left(\frac{\sqrt{1-x^2}}{n}\right) \left\{1 + \sum_{k=2}^{n-1} \frac{\sqrt{1-x^2}}{\sqrt{1-x_k^2}}\, l_k^2(x)\right\}. \end{aligned}$$

Now on using (5.10) and (5.11) we obtain

$$\begin{aligned} I_2 &\leqslant \omega_2\left(\frac{\sqrt{1-x^2}}{n}\right) \left\{1 + c_1 + c_{11}^2 \sum_{r=1}^{n} \frac{1}{r^2}\right\} \\ &\leqslant c_{15}\, \omega_2\left(\frac{\sqrt{1-x^2}}{n}\right). \end{aligned} \tag{5.14}$$

For $x = \pm 1$ (5.13) is trivially satisfied. Hence from (5.14) Lemma 5.2 follows.

6. THE PROOF OF THEOREM 3.1.

From (4.1) and (2.6) if follows that

$$\begin{aligned} A_n[f, x] - f(x) = {} & [f(1) - f(x)]\, h_1(x) + [f(-1) - f(x)]\, h_n(x) \\ & + \sum_{k=2}^{n-1} [f(x_k) - f(x)]\, h_k(x). \end{aligned} \tag{6.1}$$

On using Lemma 5.1 we at once obtain

$$|A_n[f, x] - f(x)| \leqslant (c_8 + c_9)\, \omega\left(\frac{\sqrt{1-x^2}}{n}\right) + c_{10} \sum_{r=1}^{n} \frac{1}{r^2}\, \omega\left(\frac{r \sin\theta}{n}\right).$$

Now following the same lines as in [10] we get

$$\sum_{r=1}^{n} \frac{1}{r^2}\, \omega\left(\frac{r \sin\theta}{n}\right) \leqslant \frac{c_{16}}{n} \sum_{i=1}^{n} \omega\left(\frac{\sqrt{1-x^2}}{i}\right).$$

Therefore, we obtain

$$|A_n[f, x] - f(x)| \leqslant \frac{c_{17}}{n} \sum_{i=1}^{n} \omega\left(\frac{\sqrt{1-x^2}}{i}\right).$$

This completes the proof of inequality (3.1).

Next, we shall prove (3.2). It is well known that if $f(x)$ is a polynomial of degree $\leqslant 2n-3$ (with $\mu_n'(x_k) = f'(x_k)$) then

$$B_n[f, x] \equiv f(x).$$

Since $\mu_n(x)$ is a polynomial of degree $\leqslant n$ we can write

$$\mu_n(x) = \sum_{k=1}^{n} \mu_n(x_k)\, h_k(x) + \sum_{k=2}^{n-1} \mu_n'(x_k)\, \sigma_k(x). \tag{6.2}$$

Now from (2.7), (2.8), (5.13) and (6.2) it follows that

$$\begin{aligned} &|B_n[f, x] - f(x)| \\ &\quad \leqslant |B_n[f, x] - \mu_n(x)| + |\mu_n(x) - f(x)| \\ &\quad \leqslant \sum_{k=1}^{n} |f(x_k) - \mu_n(x_k)|\, h_k(x) + |\mu_n(x) - f(x)| \\ &\quad \leqslant c_0 \sum_{k=2}^{n-1} \omega_2(\sqrt{1-x_k^2}/n)\, h_k(x) + c_0\, \omega_2(\sqrt{1-x^2}/n) \\ &\quad \leqslant (c_{15}+1)\, c_0\, \omega_2(\sqrt{1-x^2}/n) \\ &\quad \leqslant c_2\, \omega_2(\sqrt{1-x^2}/n) \end{aligned}$$

which completes the proof of Theorem 3.1.

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